Generalized Parton Distributions from Lattice QCD

Dru B. Renner
University of Arizona

Caltech
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Generalized parton distributions determine the angular momentum decomposition of the nucleon and the transverse distribution of partons within the nucleon. Additionally, in particular limits they reduce to form factors and ordinary parton distributions. I will review generalized parton distributions and present our lattice QCD calculations of moments of generalized parton distributions. In particular I will examine the transverse distribution of quarks within the nucleon and, time permitting, I will show an exploratory calculation of the nucleon axial coupling using chiral lattice fermions.

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Dru B. Renner
University of Arizona

LHPC and MILC Collaborations

http://talks.drubryantrenner.org/caltech_5-20-05.pdf

Generalized Parton Distributions from Lattice QCD

What new insight do we gain?

• Spin Decomposition - decomposition of nucleon spin into quark helicity, quark orbital, and gluon contributions

$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma_{u+d} + L_{u+d} + J_g$$

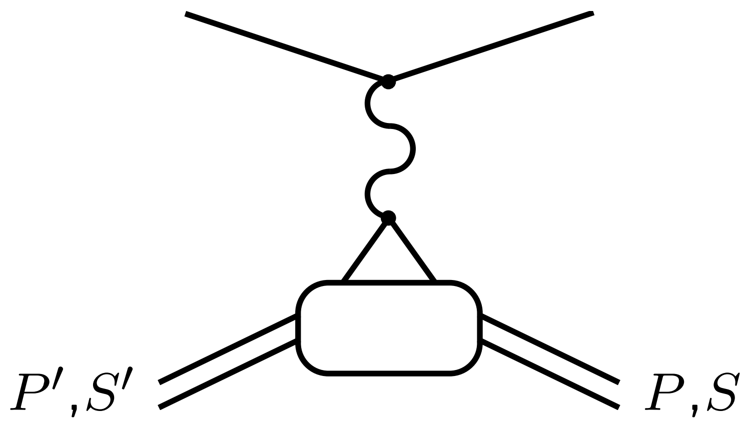
• Transverse Structure - 3D distribution of quarks in a mixed representation: 2 transverse coordinates $\vec{b}_\perp$ and 1 longitudinal momentum x

$$q(x, \vec{b}_\perp) \quad \text{and} \quad \Delta q(x, \vec{b}_\perp)$$

What experiments probe the generalized parton distributions?

Form Factors

- lepton-nucleon scattering



$$\Delta = P' - P$$
$$t = \Delta^2$$

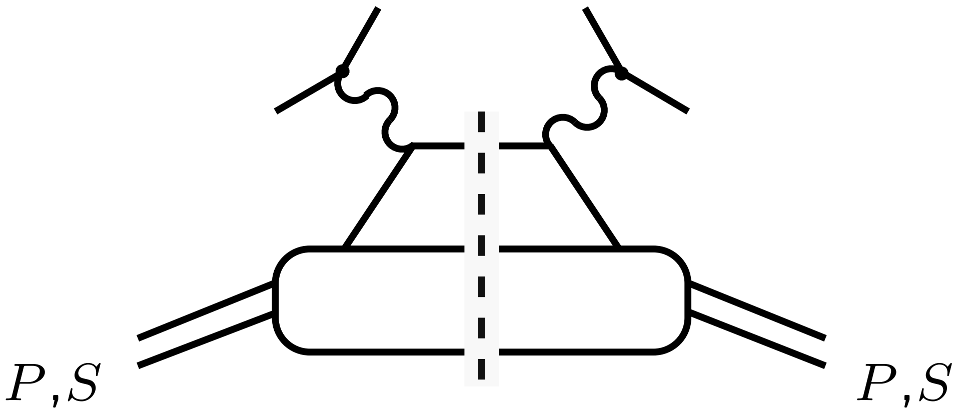
- off-forward matrix element of the electromagnetic current

$$< P', S'|J^\mu|P, S >= \overline{U}(P', S')\left(\gamma^\mu F_1(t) + \frac{i\sigma^{\mu\nu}\Delta_\nu}{2m}F_2(t)\right)U(P, S)$$

- interpretation as Fourier transform of charge and current densities *in certain limits*

Parton Distributions

- deep inelastic scattering



$$P^+ = \frac{1}{2}\left(P^t + P^z\right)$$
$$y^- = \frac{1}{2}\left(y^t - y^z\right)$$

- forward matrix element of light-cone quark correlator

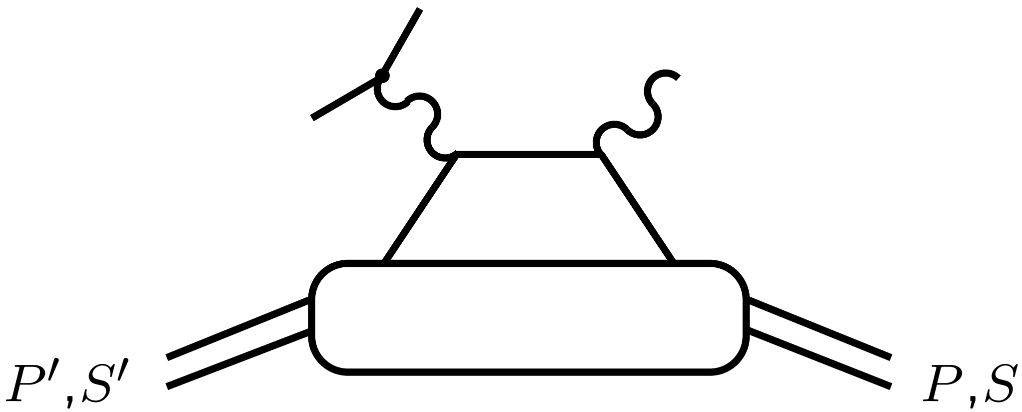
$$\mathcal{O}_q(x,\vec{b}_\perp) = \int \frac{dy^-}{4\pi} e^{ixP^+y^-}\, \overline{q}\left(-\frac{y^-}{2},\vec{b}_\perp\right) \gamma^+ q\left(\frac{y^-}{2},\vec{b}_\perp\right)$$

$$\left\langle P,S\left|\mathcal{O}_q(x,\vec{0}_\perp)\right|P,S\right\rangle = q(x)$$

- longitudinal momentum distribution in the *infinite momentum frame*

Generalized Parton Distributions

- deeply virtual Compton scattering



$$\overline{P} = \frac{1}{2}(P' + P)$$

$$\xi = -\frac{\Delta^+}{2\overline{P}^+}$$

- off-forward matrix element of light-cone quark correlator

$$\left\langle P', S' \left| \mathcal{O}_q(x, \vec{0}_\perp) \right| P, S \right\rangle =$$

$$\frac{1}{2\overline{P}^+} \overline{U}(P', S') \left(\gamma^+ H(x, \xi, t) + i \frac{\sigma^{+\nu} \Delta_\nu}{2M} E(x, \xi, t) \right) U(P, S)$$

GPD reviews: Ji hep-ph/9807358, Diehl hep-ph/0307382

What matrix elements determine the quark angular momenta and transverse quark distributions?

Generalized Form Factors

- light-cone expansion of $\mathcal{O}_q(x, \vec{b}_\perp)$ relates generalized parton distributions to generalized form factors

- unpolarized twist two operators

$$O_q^{\mu_1 \cdots \mu_n} \;=\; \overline{q} i D^{(\mu_1} \ldots i D^{\mu_{n-1}} \gamma^{\mu_n)} q$$

- off-forward matrix elements of the twist two operators [1]

$$\langle P', S' | O_q^{\mu_1 \cdots \mu_n} | P, S \rangle = \overline{U}(P', S') [\sum_{\substack{i=0 \\ \text{even}}}^{n-1} A_{ni}^q(t) K_{ni}^A(P', P)$$
$$+ \sum_{\substack{i=0 \\ \text{even}}}^{n-1} B_{ni}^q(t) K_{ni}^B(P', P) + \delta_{\text{even}}^n C_n^q(t) K_n^C(P', P)] U(P, S)$$

- similar expression for the polarized observables: $\tilde{A}_{ni}^q(t)$ and $\tilde{B}_{ni}^q(t)$

[1] X. D. Ji hep-ph/9807358

Basic Properties of Generalized Form Factors

- moments of parton distributions - $\langle P|O_q^{\mu_1\cdots\mu_n}|P\rangle$ and $\langle P|\tilde{O}_q^{\mu_1\cdots\mu_n}|P\rangle$

$$A_{n0}^q(0) = \int_{-1}^{1} dx\ x^{n-1}\, q(x) \quad \text{and} \quad \tilde{A}_{n0}^q(0) = \int_{-1}^{1} dx\ x^{n-1}\, \Delta q(x)$$

- form factors - $\mathcal{O}_q^\mu = \overline{q}\gamma^\mu q$ and $\tilde{\mathcal{O}}_q^\mu = \overline{q}\gamma^\mu\gamma^5 q$

$$A_{10}^q(t) = F_1^q(t) \quad \text{and} \quad B_{10}^q(t) = F_2^q(t)$$

$$\tilde{A}_{10}^q(t) = G_A^q(t) \quad \text{and} \quad \tilde{B}_{10}^q(t) = G_P^q(t)$$

Quark Angular Momenta and Transverse Quark Distributions

- quark angular momenta [1]

$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma^{u+d} + L^{u+d} + J^g$$

$$\Delta\Sigma^q = \tilde{A}^q_{10}(0) \qquad J^q = \frac{1}{2}\left(A^q_{20}(0) + B^q_{20}(0)\right) \qquad L^q = J^q - \frac{1}{2}\Delta\Sigma^q$$

- transverse quark distributions [2]

$$\int_{-1}^{1} dx\; x^{n-1}\, q(x, \vec{b}_\perp) \;=\; \int \frac{d^2\Delta_\perp}{(2\pi)^2} e^{-i\vec{b}_\perp\cdot\vec{\Delta}_\perp} A^q_{n0}(-\vec{\Delta}^2_\perp)$$

$$\int_{-1}^{1} dx\; x^{n-1}\, \Delta q(x, \vec{b}_\perp) \;=\; \int \frac{d^2\Delta_\perp}{(2\pi)^2} e^{-i\vec{b}_\perp\cdot\vec{\Delta}_\perp} \tilde{A}^q_{n0}(-\vec{\Delta}^2_\perp)$$

[1] X. D. Ji hep-ph/9603249
[2] M. Burkardt hep-ph/0005108

How do we calculate the matrix elements
of twist two operators in lattice QCD?

Lattice Fields and Correlation Functions

- Grassmann fermion fields $\psi_\alpha^a(x)$ and $\overline{\psi}_\alpha^a(x)$ at every lattice point x
- SU(3) gauge fields $U_\mu^{ab}(x)$ at every lattice link $x \to x+\mu$
- correlation function of $2N$ fermion fields and M gauge links

$$\left\langle \psi_1 \cdots \psi_N \overline{\psi}_N \cdots \overline{\psi}_1 U_1 \cdots U_M \right\rangle =$$

$$= \int DU \int D\psi D\overline{\psi}\ e^{-S_G[U]} e^{-\overline{\psi} M[U] \psi} \psi_1 \cdots \psi_N \overline{\psi}_N \cdots \overline{\psi}_1 U_1 \cdots U_M$$

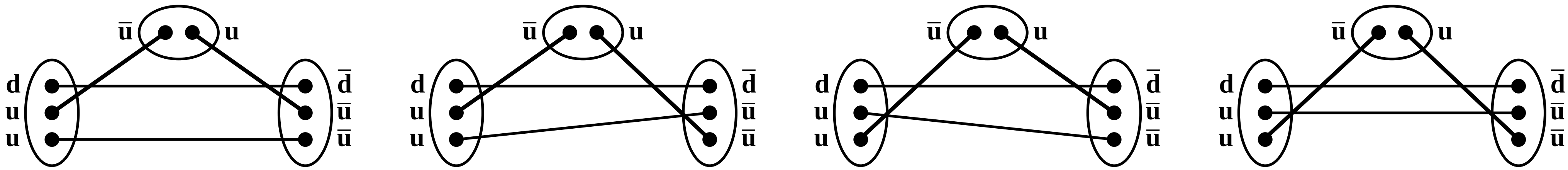
$$= \int DU\ e^{-S_G[U]}\ \det{}^{N_f}(M[U])\ U_1 \cdots U_M\ \sum_\pi (-1)^\pi M[U]^{-1}_{1\pi_1} \cdots M[U]^{-1}_{N\pi_N}$$

- numerical integration

$$\int DU\ e^{-S_G[U]}\ \det{}^{N_f}(M[U])\ F[U] = \frac{1}{N} \sum_{i=1}^{N} F[U^i] \pm \frac{\sigma_F}{\sqrt{N}}$$

Quark Diagrams

- connected u quark diagrams



- disconnected u diagrams - not calculated - $(u + d)$ matrix elements only

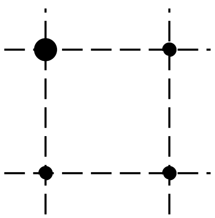
- connected d quark diagrams

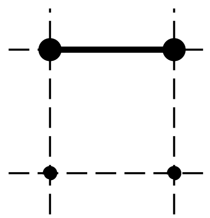
- disconnected d diagrams - not calculated - $(u + d)$ matrix elements only

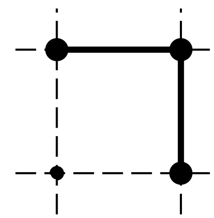
Building Blocks

- twist two operators $\overline{q}\Gamma D^{\mu_1}\cdots D^{\mu_n}q$ can be written in terms of a basic set of building blocks

$\overline{q}(x)\,\Gamma_i\,q(x)$ $\qquad$ $\overline{q}(x+\widehat{\mu})\,\Gamma_i\,U^\dagger_\mu(x)\,q(x)$ $\qquad$ $\overline{q}(x+\widehat{\nu}+\widehat{\mu})\,\Gamma_i\,U^\dagger_\nu(x+\widehat{\mu})U^\dagger_\mu(x)\,q(x)$







- Γ_i $(i=1,\ldots,16)$ denote a complete basis of 4×4 Dirac spin matrices
- calculate upto 3 gauge link insertions: $U^\dagger_{\mu_n}(x+\widehat{\mu}_{n-1}+\cdots\widehat{\mu}_1)\cdots U^\dagger_{\mu_1}(x)$
- calculate a broad range of momentum transfers $\vec{q}$
- more than $380,000$ correlation functions

Overdetermined Set of Lattice Observables

- i labels all combinations of operator indices $(\mu_1, \ldots, \mu_n)$ and momenta P', P which give the same $t = (P' - P)^2$

$$\mathcal{O}_i^{\overline{\mathrm{MS}}} = \left\langle P' | \mathcal{O}^{\mu_1 \cdots \mu_n} | P \right\rangle$$

- expand $\mathcal{O}_i^{\overline{\mathrm{MS}}}$ in terms of generalized form factors, $F_\alpha = A_{ni}$, B_{ni}, ...

$$\mathcal{O}_i^{\overline{\mathrm{MS}}} = \sum_{\alpha=1}^{M} K_{i\alpha} F_\alpha(t)$$

- match $\mathcal{O}_i^{\overline{\mathrm{MS}}}$ to $\mathcal{O}_i^{\mathrm{lat}}$ with one loop matching coefficients Z_{ij}

$$\mathcal{O}_i^{\overline{\mathrm{MS}}} = \sum_{j=1}^{N} Z_{ij} \mathcal{O}_j^{\mathrm{lat}}$$

- overdetermined set of observables, $N > M$

$$\mathcal{O}_i^{\mathrm{lat}} = \sum_\alpha (Z^{-1}K)_{i\alpha} F_\alpha(t) \qquad \chi^2 = \sum_{i=1}^{N} \left(\frac{\sum_{\alpha=1}^{M} (Z^{-1}K)_{i\alpha} F_\alpha(t) - \mathcal{O}_i^{\mathrm{lat}}}{\sigma_i} \right)^2$$

What do we learn about the quark angular momentum and transverse quark distributions?

Calculations with Heavy-*ish* Quarks

- Wilson quarks
- flavors: $N_F = 2$
- lattice spacing: $a = 0.095$ fm
- lattice size: $L = 1.52$ fm
- pion masses: $M_\pi = 753(10)$, $835(13)$, $895(15)$ MeV

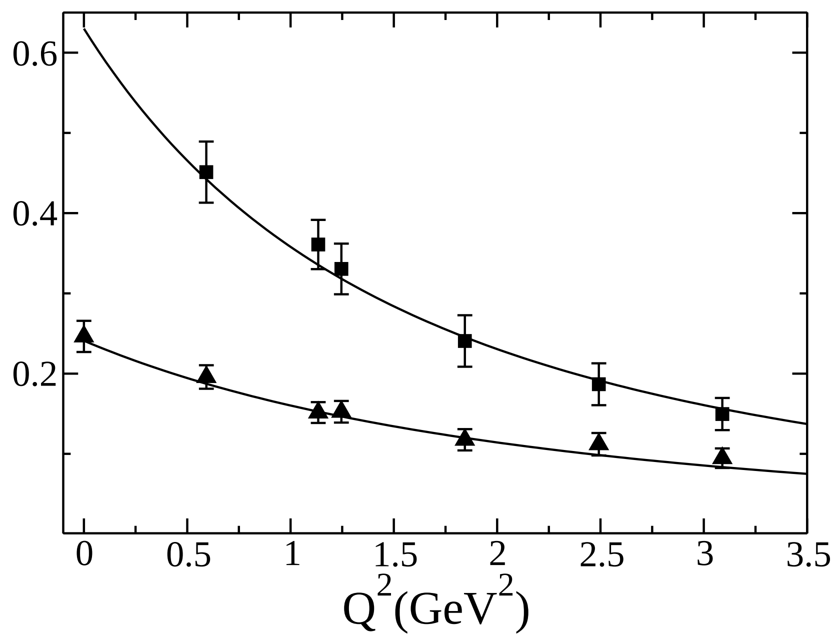
What do we learn about the nature of the nucleon spin?

Quark Angular Momenta

$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma^{u+d} + L^{u+d} + J^g$$

$$\Delta\Sigma_q = \frac{1}{2}\tilde{A}^q_{10}(0) \qquad J_q = \frac{1}{2}\left(A^q_{20}(0) + B^q_{20}(0)\right)$$

$$L_q = J_q - \frac{1}{2}\Delta\Sigma_q \qquad J_g = \frac{1}{2} - J_{u+d}$$



- large N_C: $B^{u-d}_{ni}/A^{u-d}_{ni} \sim N_C$

Quark Angular Momenta: $m_\pi = 895$ MeV

- quark helicity, quark orbital, and gluon contributions (modulo disconnected diagrams)

$$2 \cdot \tfrac{1}{2} = \quad \Delta\Sigma^{u+d} \quad + \quad 2L^{u+d} \quad + \quad 2J^{g}$$

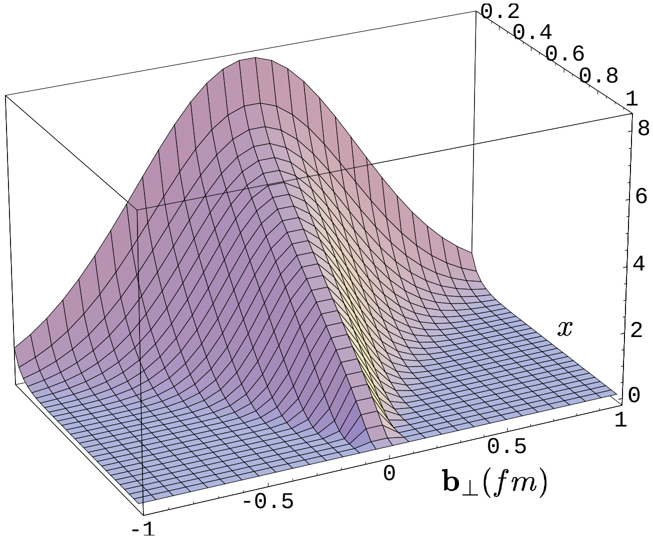
$$+0.682(20) \qquad -0.002(3) \qquad +0.320(16)$$

- quark orbital motion, $L^{u-d} = -0.193(32)$

$$2L^{q}_{\text{min}} = 2 \cdot \left(\frac{|L^u| + |L^d|}{2} \right) \geq |L^{u-d}| = 0.193(32)$$

What do we learn about the transverse quark structure of the nucleon?

Transverse Distributions



$$A^q_{n0}(-\vec{\Delta}^2_\perp) = \int d^2 b_\perp \; e^{i\vec{\Delta}_\perp \cdot \vec{b}_\perp} \int_{-1}^{1} dx \; x^{n-1} \, q(x, \vec{b}_\perp)$$

$$\langle b^2_\perp \rangle^q_{(n)} = -4 \, \frac{A^{q\,\prime}_{n0}(0)}{A^q_{n0}(0)}$$

- at $x = 1$ a single quark carries all the momentum

$$\lim_{x \to 1} q(x, \vec{b}_\perp) \propto \delta^2(\vec{b}_\perp)$$

- higher moments A^q_{n0} weight $x \sim 1$ more heavily

$$\lim_{n \to \infty} A^q_{n0}(t) \propto \int d^2 b_\perp \; e^{i\vec{\Delta}_\perp \cdot \vec{b}_\perp} \delta^2(\vec{b}_\perp) = \text{constant}$$

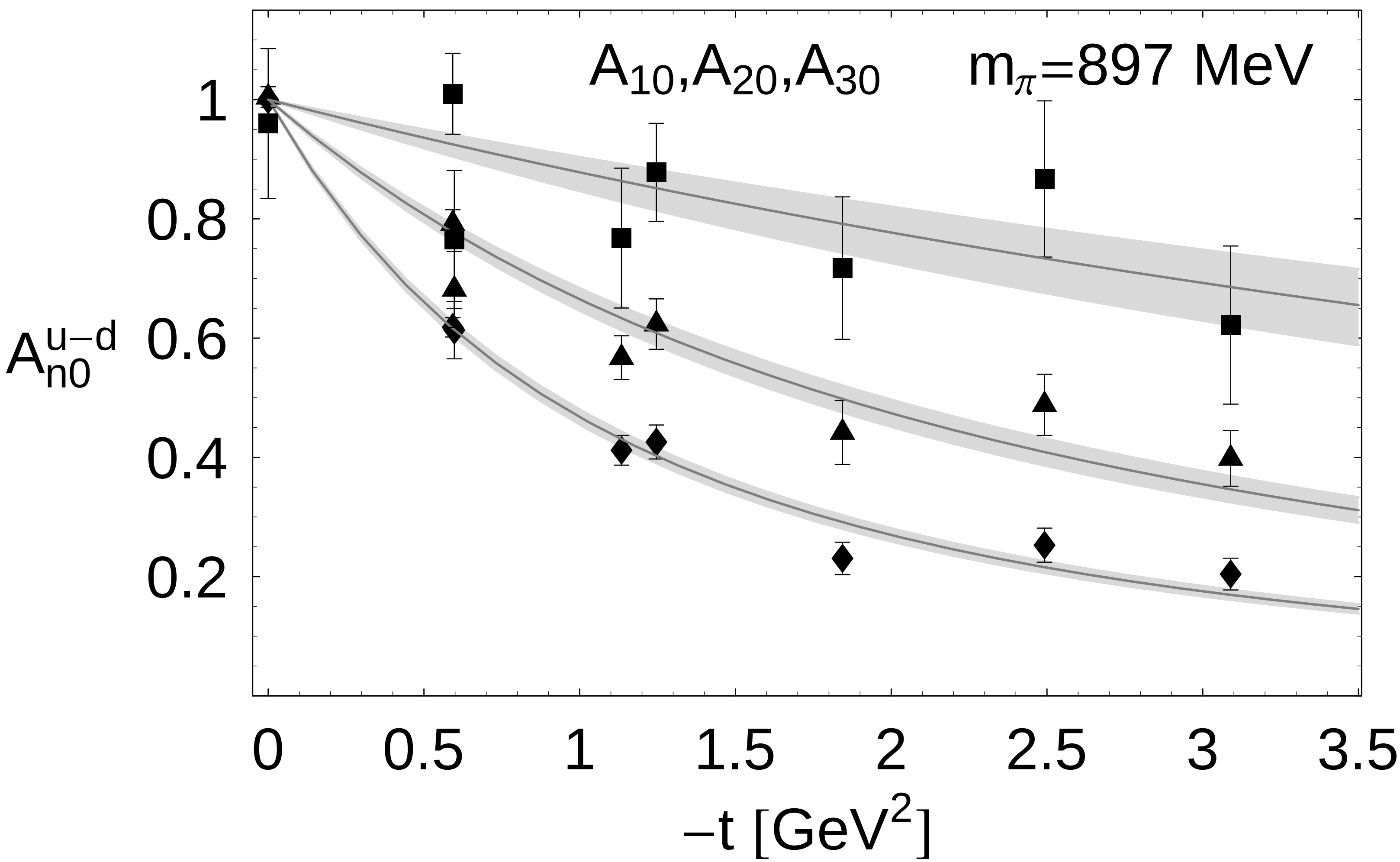
- slopes of A^q_{n0} should decrease as n increases

- A_{10}, A_{30}, $\tilde{A}_{20}$ measure $q - \overline{q}$ & $\tilde{A}_{10}$, $\tilde{A}_{30}$, A_{20} measure $q + \overline{q}$

graph from M. Burkardt hep-ph/0207047

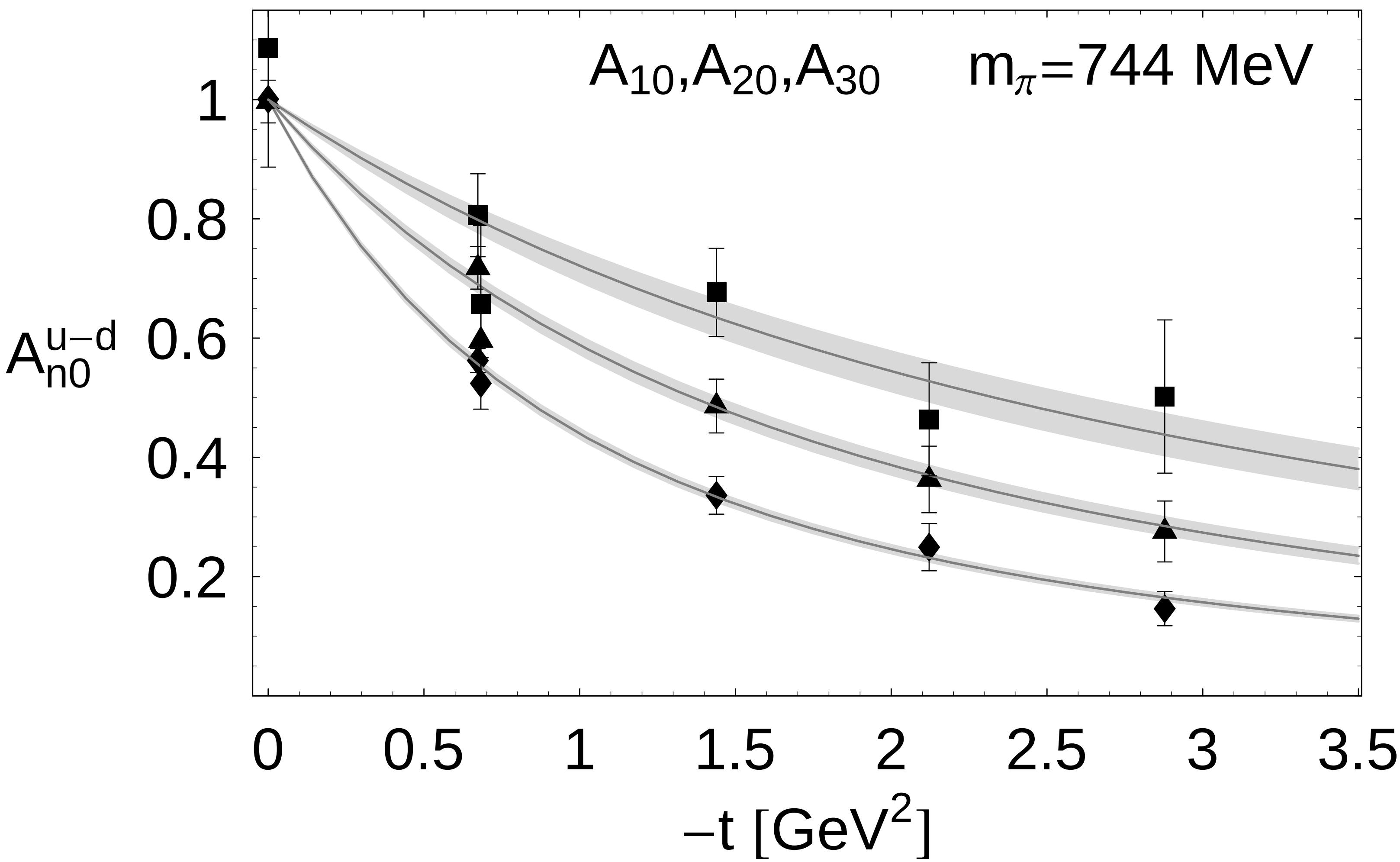
Transverse Distributions: $m_\pi = 897$ MeV

- slope of $A_{10}^{u-d} = -0.93 \pm 0.04$ (GeV)$^{-2}$
- slope of $A_{30}^{u-d} = -0.13 \pm 0.03$ (GeV)$^{-2}$ (factor of 7)



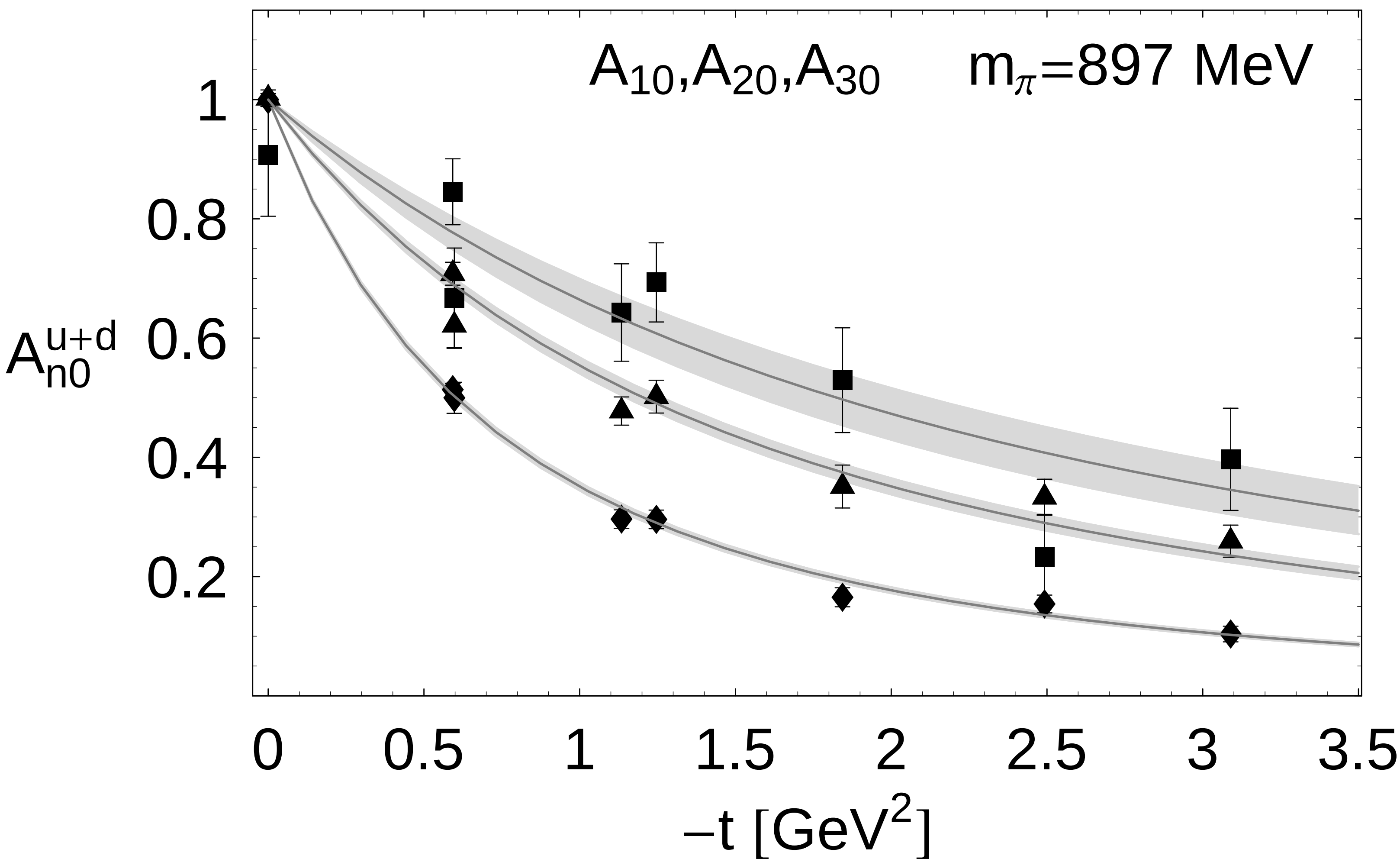
Transverse Distributions: Mass Dependence

- slope of $A_{10}^{u-d} = -1.02 \pm 0.03\ (\mathrm{GeV})^{-2}$
- slope of $A_{30}^{u-d} = -0.36 \pm 0.04\ (\mathrm{GeV})^{-2}$ (factor of 3)



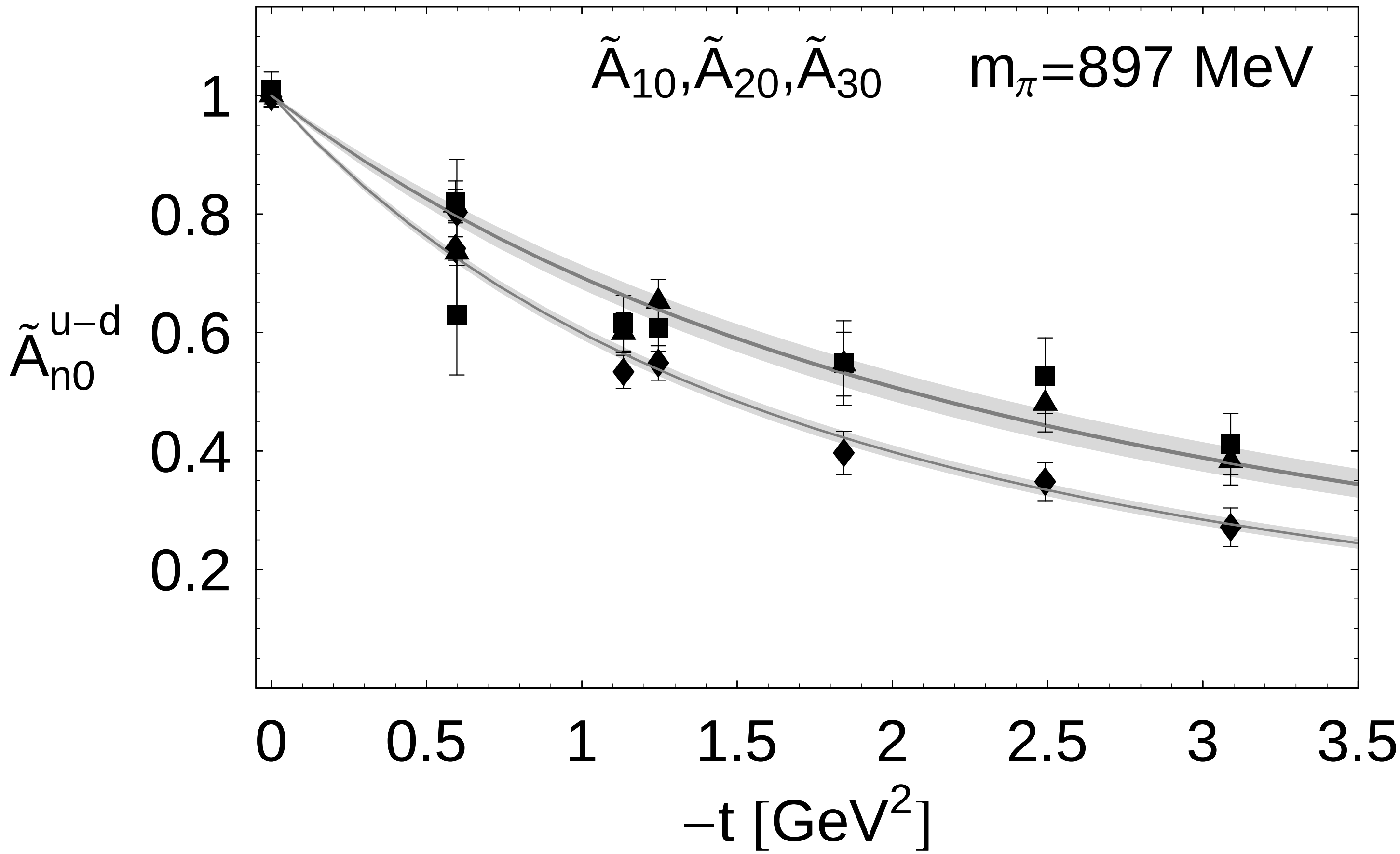
Transverse Distributions: Flavor Dependence

- slope of $A_{10}^{u+d} = -1.38 \pm 0.03\ (\mathrm{GeV})^{-2}$
- slope of $A_{30}^{u+d} = -0.45 \pm 0.07\ (\mathrm{GeV})^{-2}$ (factor of 3)



Transverse Distributions: Spin Dependence

- slope of $\tilde{A}_{10}^{u-d} = -0.58 \pm 0.02\ (\text{GeV})^{-2}$
- slope of $\tilde{A}_{30}^{u-d} = -0.40 \pm 0.03\ (\text{GeV})^{-2}$ (factor of 1.5)



- helicity retention: as $x \to 1$ so does $\Sigma \to 1$

Transverse Distributions: x Dependence

- transverse rms radius

$$\left\langle b_\perp^2 \right\rangle_x = \frac{\int d^2b_\perp \, b_\perp^2 \, q(x,\vec{b}_\perp)}{\int d^2b_\perp \, q(x,\vec{b}_\perp)}$$

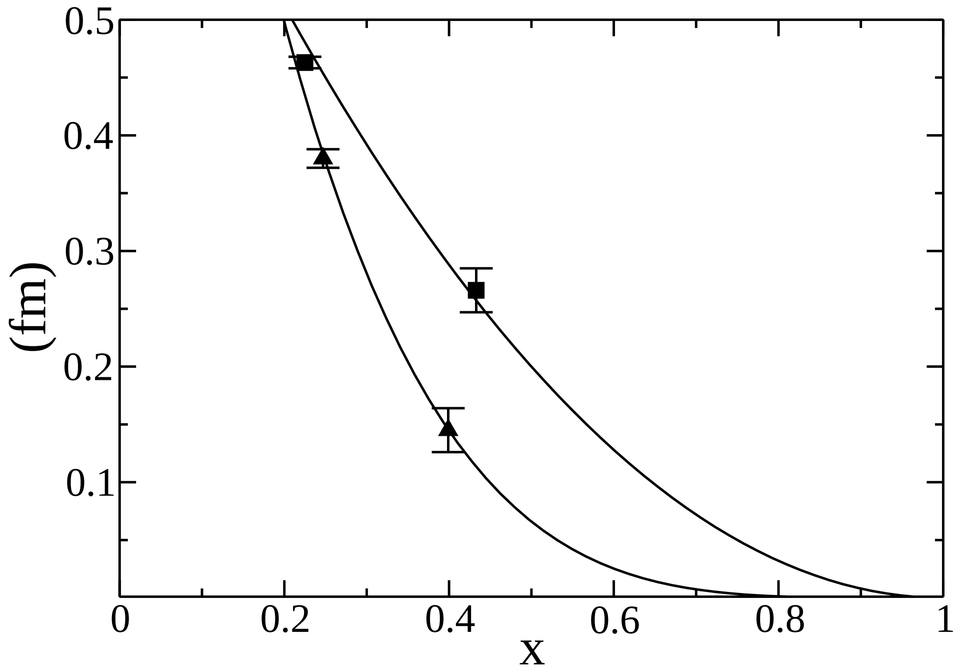
- transverse rms *moment* radius

$$\frac{A'_{n0}(0)}{A_{n0}(0)} = -\frac{1}{4}\left\langle b_\perp^2 \right\rangle_{(n)} \qquad \left\langle b_\perp^2 \right\rangle_{(n)} = \frac{\int d^2b_\perp \, b_\perp^2 \, \int_{-1}^1 dx \, x^{n-1} q(x,\vec{b}_\perp)}{\int d^2b_\perp \, \int_{-1}^1 dx \, x^{n-1} q(x,\vec{b}_\perp)}$$

- the average x in $\left\langle b_\perp^2 \right\rangle_{(n)}$

$$x_{\text{av}}^{(n)} = \frac{\int d^2b_\perp \, \int_{-1}^1 dx \, |x| \cdot x^{n-1} q(x,\vec{b}_\perp)}{\int d^2b_\perp \, \int_{-1}^1 dx \, x^{n-1} q(x,\vec{b}_\perp)} \approx \frac{< x^n >}{< x^{n-1} >}$$

Transverse Distributions: x Dependence

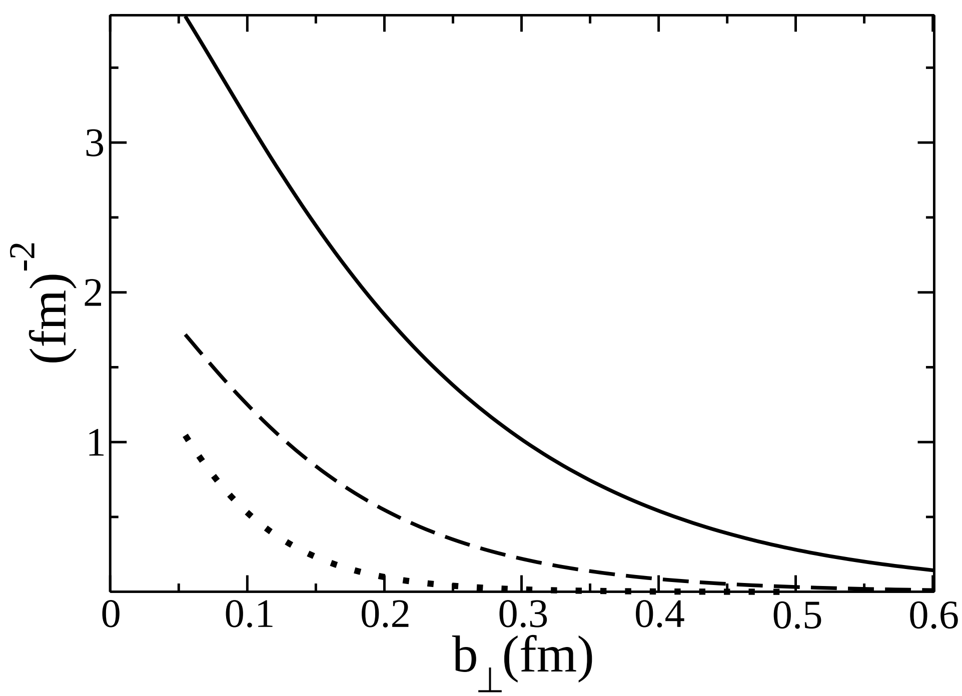


- non-singlet radius: $\sqrt{\left\langle b_\perp^2 \right\rangle_{u-d}^{(1)}} = 0.38$ fm & $\sqrt{\left\langle b_\perp^2 \right\rangle_{u-d}^{(3)}} = 0.15$ fm, 61% decrease
- singlet radius: $\sqrt{\left\langle b_\perp^2 \right\rangle_{u+d}^{(1)}} = 0.46$ fm & $\sqrt{\left\langle b_\perp^2 \right\rangle_{u+d}^{(3)}} = 0.27$ fm, 41% decrease

Transverse Distributions: $\vec{b}_\perp$ Dependence

$$q_1(\vec{b}_\perp) = \int_{-1}^{1} dx\ q(x, \vec{b}_\perp) = \int \frac{d^2\Delta_\perp}{(2\pi)^2} e^{-i\vec{b}_\perp \cdot \vec{\Delta}_\perp} A_{10}^q(-\vec{\Delta}_\perp^2)$$

$$q_2(\vec{b}_\perp) = \int_{-1}^{1} dx\ x\, q(x, \vec{b}_\perp) = \int \frac{d^2\Delta_\perp}{(2\pi)^2} e^{-i\vec{b}_\perp \cdot \vec{\Delta}_\perp} A_{20}^q(-\vec{\Delta}_\perp^2)$$



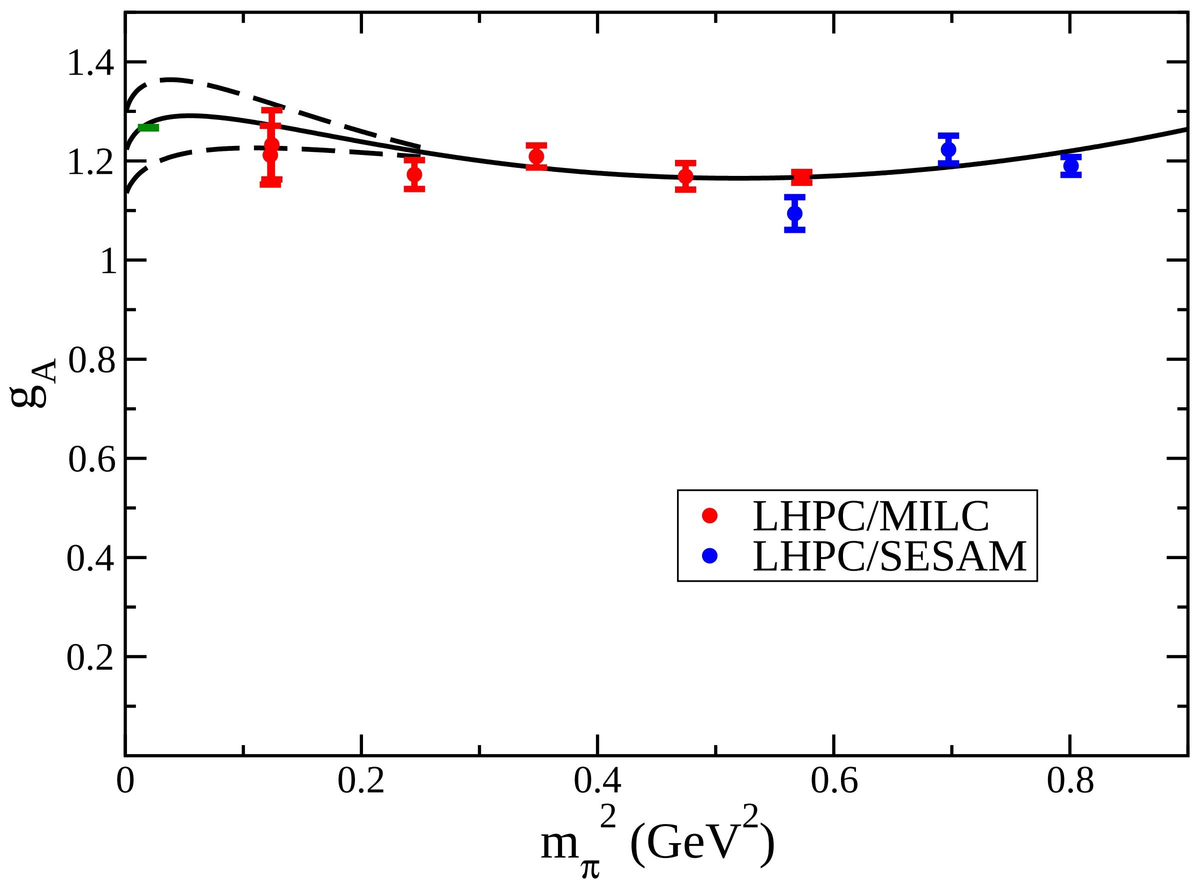
What about the chiral limit?

Calculations with Light-*ish* Quarks

- Kogut-Susskind sea quarks and domain wall valence quarks
- flavors: $N_F = 2 + 1$
- lattice spacing: $a = 0.13$ fm
- lattice sizes: $L = 2.6$ fm and $L = 3.6$ fm
- pion masses: $M_\pi =$ 329(15), 354(3), 564(2), 693(3), 730(3) MeV

Axial Charge g_A

- curve is one loop chiral perturbation theory (including Δ) [1]
- parameters: f_π, g^A_{NN}, $g^A_{N\Delta}$, $g^A_{\Delta\Delta}$, $m_\Delta - m_N$, $B_9 - g_A B_{20}$
- only g^A_{NN} and $g^A_{\Delta\Delta}$ are fit below



[1] Hemmert et al hep-lat/0303002

Conclusions

- unambiguous observation of quark orbital motion, $L^{u-d} = -0.193(32)$, for even heavy pion masses

- the transverse size of the nucleon, $\sqrt{\langle r_\perp^2 \rangle}$, in the heavy pion world, shows a significant dependence on the longitudinal momentum fraction $\langle x \rangle$

- current and ongoing calculations are exploring the chiral limit of these and other observables, for example g_A

Extra Slides

Moments of Generalized Parton Distributions

- moments of generalized parton distributions

$$\int_{-1}^{1} dx\ x^{n-1} H_q(x,\xi,t) = \sum_{\substack{i=0\\ \text{even}}}^{n-1} A^q_{ni}(t)(-2\xi)^i + \delta^n_{\text{even}} C^q_n(t)(-2\xi)^n$$

$$\int_{-1}^{1} dx\ x^{n-1} E_q(x,\xi,t) = \sum_{\substack{i=0\\ \text{even}}}^{n-1} B^q_{ni}(t)(-2\xi)^i - \delta^n_{\text{even}} C^q_n(t)(-2\xi)^n$$

- transverse momentum transfer, $\xi \to 0$

$$\int_{-1}^{1} dx\ x^{n-1} H_q(x,0,t) = A^q_{n0}(t)$$

$$\int_{-1}^{1} dx\ x^{n-1} E_q(x,0,t) = B^q_{n0}(t)$$

- similar results relating the polarized GPDs, $\tilde{H}_q(x,0,t)$ and $\tilde{E}_q(x,0,t)$, to the polarized GFFs, $\tilde{A}^q_{n0}(t)$ and $\tilde{B}^q_{n0}(t)$

Quark Angular Momentum

- angular momentum operator

$$J_q^i = \frac{1}{2}\epsilon^{ijk} \int \, d^3x \left(T_q^{0k} x^j - T_q^{0j} x^k \right)$$

- energy momentum tensor

$$T_q^{\mu\nu} = \overline{q}\gamma^{\{\mu} i D^{\nu\}} q = O_q^{\mu\nu}$$

- angular momentum

$$J_q = \langle P, 1/2 | J_q^z | P, 1/2 \rangle \quad \text{involves} \quad \langle P, 1/2 | O_q^{\mu\nu} | P, 1/2 \rangle$$

- matrix elements of $n = 2$ twist 2 operator $O_q^{\mu\nu}$: $A_{20}(t)$, $B_{20}(t)$, $C_2(t)$

$$J_q = \frac{1}{2} \left(A_{20}^q(0) + B_{20}^q(0) \right)$$

Transverse Quark Distribution

- wave packet in transverse coordinates

$$|\psi\rangle = \int \frac{d^2p_\perp}{(2\pi)} \frac{\psi(\vec{p}_\perp)}{\sqrt{2E_{\vec{p}}}} |\vec{p}_\perp, P_z\rangle \qquad \vec{p} = (\vec{p}_\perp, P_z)$$

- transverse quark distribution

$$q_\psi(x, \vec{b}_\perp) = \langle\psi|O_q(x, \vec{b}_\perp)|\psi\rangle \quad \text{involves} \quad \langle\vec{k}_\perp, P_z|O_q(x, \vec{b}_\perp)|\vec{p}_\perp, P_z\rangle$$

- infinite momentum limit & transverse localization

$$M \ll P_z \qquad \psi(\vec{k}_\perp) \approx \psi(\vec{p}_\perp)$$

- matrix elements of $O_q(x, \vec{b}_\perp)$: $H_q(x, \xi, t)$, $E_q(x\ , \xi, t)$

$$q(x, \vec{b}_\perp) = \int d^2\Delta_\perp \, e^{-i\vec{b}_\perp \cdot \vec{\Delta}_\perp} H_q(x, 0, -\vec{\Delta}_\perp^2)$$

- relativistic corrections controlled by

$$\frac{1}{\sqrt{M^2 + P_z^2}} \quad \text{not} \quad \frac{1}{M}$$